Published by Institute of Physics Publishing for SISSA
Received: *November 13, 2007*
Accepted: *January 23, 2008*
Published: *February 14, 2008*

Half BPS states in $AdS_5 \times Y^{p,q}$

Edi Gava,[abc] **Giuseppe Milanesi,**[bcd] **K.S. Narain**[a] **and Martin O'Loughlin**[e]

[a] *High Energy Section, The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy*
[b] *Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, 34014 Trieste, Italy*
[c] *Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2-4, 34014 Trieste, Italy*
[d] *Institut für Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland*
[e] *University of Nova Gorica, Vipavska 13, 5000, Nova Gorica, Slovenia*
E-mail: gava@ictp.it, milanesi@itp.phys.ethz.ch, narain@ictp.it, Martin.John.O.Loughlin@p-ng.si

Abstract: We study a class of solutions of IIB supergravity which are asymptotically $AdS_5 \times Y^{p,q}$. They have an $\mathbb{R} \times \mathrm{SO}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ isometry and preserve half of the 8 supercharges of the background geometry. They are described by a set of second order differential equations that we have found and analysed in a previous paper, where we studied 1/8 BPS states in the maximally supersymmetric $AdS_5 \times S^5$ background. These geometries correspond to certain chiral primary operators of the $\mathcal{N} = 1$ superconformal quiver theories, dual to IIB theory on $AdS_5 \times Y^{p,q}$.
We also show how to recover the $AdS_5 \times Y^{p,q}$ backgrounds by suitably doubling the number of preserved supersymmetries. We then solve the differential equations perturbatively in a large AdS_5 radius expansion, imposing asymptotic $AdS_5 \times Y^{p,q}$ boundary conditions. We compute the global baryonic and mesonic charges, including the R-charge. As for the computation of the mass, i.e. the conformal dimension Δ of the dual field theory operators, which is notoriously subtle in asymptotically AdS backgrounds, we adopt the general formalism due to Wald and collaborators, which gives a finite result, and verify the relation $\Delta = 3R/2$, demanded by the $\mathcal{N} = 1$ superconformal algebra.

Keywords: AdS-CFT Correspondence, Supergravity Models, Supersymmetric gauge theory.

JHEP02(2008)050

Contents

1. Introduction

One of the most impressive checks of the AdS/CFT has been obtained a few years ago [1], where a very precise correspondence between supergravity geometries and states in the dual SU(N) $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{R} \times S^3$ has been established at the 1/2 BPS level. More precisely, the free-fermion picture arising in the large N gauge theory reduced on S^3 and restricted to the 1/2 BPS sector, has been shown to appear quite precisely in the exact solution of the 1/2 BPS geometries on the supergravity side. This goes beyond the giant graviton regime, which corresponds to probe D3 branes wrapped on S^3's either in AdS_5 or S^5 [2−5] , in the sense that it captures the full gravitational backreacted geometry. Attempts to generalise this picture to less supersymmetric geometries/states appeared recently in [6−12]. An important class of non-local normalizable states (Wilson lines) and the corresponding dual geometries were studied in [13, 14]

Of course, another, but related, direction to explore would be to consider BPS states in less supersymmetric bulk theories. Interesting examples are the dual pairs given by string theory on $AdS_5 \times Y^{p,q}$ and certain $\mathcal{N}=1$ Superconformal Quiver Gauge Theories, which have been subject of intense study recently. In [15, 16] the explicit metric on a class of Sasaki Einstein manifolds $Y^{p,q}$ was constructed. A direct generalisation of the AdS/CFT correspondence relates Type IIB String Theory on $AdS_5 \times Y^{p,q}$, with $\mathcal{N}=1$ Quiver Gauge Theories [17]. The parameters are identified as follows

$$\frac{L^2_{\rm AdS}}{4\pi\ell_s^2} = \left(\frac{\lambda}{4\pi}\frac{\pi^3}{Vol(Y^{p,q})}\right)^{\frac{1}{2}} \qquad g_s = \frac{\lambda}{N}\,. \tag{1.1}$$

Every $Y^{p,q}$ manifold has an $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ isometry group and the $AdS_5 \times Y^{p,q}$ solutions preserve 8 of the original 32 supersymmetries of type IIB supergravity. Supersymmetric branes wrapping cycles in $Y^{p,q}$ have been analysed in the probe approximation in [18, 19] and they may be considered as generalisations of giant gravitons. Dual giant gravitons were studied in [21, 22]. A distinguishing feature of the $Y^{p,q}$ manifolds, unlike S^5, is the presence of a non-trivial 3-cycle. D3-branes can thus wrap such a non trivial cycle and be stable: such branes are dual to baryons in the gauge theory, the so called dibaryons, which are built out of products of N chiral superfields [20]. Correspondingly, on the supergravity side there is a gauge field coming from the four-form Ramond-Ramond gauge field, which is dual to the baryonic current of the Gauge Theroy.

In the quiver gauge theories associated to $Y^{p,q}$ manifolds, there are $2p$ $\mathrm{SU}(N)$ gauge groups and 4 types of chiral superfields, X, Y, U_i and V_i, $i=1,2$ in the bifundamental of $\mathrm{SU}(N)\times\mathrm{SU}(N)$, with the precise gauge assignments encoded in the quiver diagram. The fields U and V are furthermore doublets of an SU(2) flavour symmetry. With a generic superfield A^{β}_{α}, $\alpha \in \mathbf{N}$ and $\beta \in \bar{\mathbf{N}}$, in the bifundamental of $\mathrm{SU}(N)\times\mathrm{SU}(N)$, one can construct dibaryonic gauge singlets $\epsilon_{\alpha_1,\ldots,\alpha_N}\epsilon^{\beta_1,\ldots,\beta_N}A^{\alpha_1}_{\beta_1}\cdots A^{\alpha_N}_{\beta_N}$ The dibaryons constructed with the SU(2) doublets U_i and V_i are furthermore in the $N+1$ dimensional representation of SU(2). In addition to baryonic-like operators one can construct also mesonic-like operators, which are neutral under the baryonic charge. These are the precise analogs of giant gravitons of the $\mathcal{N}=4$ theory. In any case, since our geometries preserve an SU(2), in addition to $\mathbb{R}\times\mathrm{SO}(4)\times\mathrm{U}(1)$, they correspond to SU(2) singlet operators on the gauge theory side, e.g. those constructed with the chiral superfields X and Y. The three U(1) charges, i.e. the R-charge, a flavour U(1) and the baryonic charge, will appear as integration constants in our asymptotic solutions.

In [12] solutions of the type IIB equations of motion with non trivial R-R 5-form and $\mathbb{R}\times\mathrm{SO}(4)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry group preserving 4 supercharges have been studied. $AdS_5\times Y^{p,q}$ geometries are clearly contained in this class: the $\mathbb{R}\times\mathrm{SO}(4)$ is the non compact version of $\mathrm{U}(1)\times\mathrm{SO}(4)\subset\mathrm{SO}(2,4)$, while the $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ isometry group of $Y^{p,q}$ is contained in the generic $\mathrm{SU}(2)\times\mathrm{U}(1)$ bosonic symmetry.

In this paper we first show in detail how to recover the $AdS_5\times Y^{p,q}$ geometries from the generic solutions studied in [12] by requiring that additional 4 supercharges be preserved. We then study 1/2 BPS excitations of such geometries, namely generic 1/8 BPS solutions of type IIB supergravity with $AdS_5\times Y^{p,q}$ asymptotics and $\mathbb{R}\times\mathrm{SO}(4)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry: they represent an expansion of the fully backreacted geometries of D3 branes in $AdS_5\times Y^{p,q}$. The brane source is substituted by flux in the same spirit as in the original [1].

JHEP02(2008)050

Such geometries carry three net global U(1) charges which are dual to the R-charge, a U(1) flavour charge and the baryonic charge of the gauge theory. They are determined by four scalar functions defined on a halfspace which solve four nonlinear coupled differential equations. In order to specify the asymptotics and charges of the solutions we solve such equations perturbatively at large AdS_5 radius. The zeroth order fixes the metric and the RR 5-form as needed to describe correctly the $AdS_5 \times Y^{p,q}$ geometries, the first subleading corrections determine the aforementioned global U(1) charge and the second subleading correction is necessary to obtain the value of the mass. Solutions which carry only R-charge have been studied in [23] at the linearised level.

The definition of mass is somewhat subtle in asymptotically AdS spacetimes, [24, 25] but it is even subtler when one is dealing with states in asymptotically $AdS_5 \times X^5$, with compact X^5, due to the fact that the subleading terms in the metric, that in principle can be used to determine the mass, mix the AdS_5 and M_5 coordinates. We deal with this problem by adopting a 10-dimensional version of the general construction of [26], to find the conserved Hamiltonian and thus the correct definition of the mass. We then determine the mass of our states and check that the BPS condition, relating the mass to the R-charge, is indeed satisfied by our asymptotic solutions.

The paper is organised as follows. In section 2 we give a brief summary of the results of [12]. In section 3 we show how to obtain the $AdS_5 \times Y^{p,q}$ geometries from the general solutions. In section 4 we solve the system of differential equations up to second order in large AdS_5 radius (the details of the second order solutions are showed in appendix A). In section 5 we show how to obtain the R charge and the U(1) flavour charge of the solutions. In section 6 we discuss subleading corrections to the RR 5-form and derive the baryon charge of the solutions. In section 7 we discuss how to correctly define the mass for a space-time which is asymptotically a product with an AdS_5 factor. Finally, in section 8 we present some conclusions.

2. Description of 1/8 BPS states

Generic solutions of type IIB Supergravity preserving 4 of the 32 supersymmetries of the theory and an $\mathbb{R} \times \mathrm{SO}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ bosonic symmetry have been constructed perturbatively in [12]. The metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_i \mathrm{d}x^i)^2 + h^2 \frac{\rho_1^2}{\rho_3^2}(T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2) + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2 + \\ + \rho_1^2\big((\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\big) + \rho_3^2(\sigma^{\hat{3}} - A_t \mathrm{d}t - A_i \mathrm{d}x^i)^2 \end{aligned} \tag{2.1}$$

with $i = 1, 2$; the coordinate y is the product of two of the radii,

$$y = \rho_1 \tilde{\rho} > 0 \,. \tag{2.2}$$

and the function h is given by

$$h^{-2} = \tilde{\rho}^2 + \rho_3^2(1 + A_t)^2 \,. \tag{2.3}$$

The space is a fibration of a squashed 3-sphere (on which the SU(2) left-invariant 1-forms $\sigma^{\hat{a}}$ are defined) and a round 3-sphere $\tilde{\Omega}_3$ (on which the SU(2) left-invariant 1-forms $\sigma^{\tilde{a}}$ are defined) over a four dimensional manifold.

The left invariant 1-forms are given by:

$$\sigma^{\hat{1}} = -\frac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) \qquad \sigma^{\tilde{1}} = -\frac{1}{2}(\cos\tilde{\psi}\, d\tilde{\theta} + \sin\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \tag{2.4}$$

$$\sigma^{\hat{2}} = -\frac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) \qquad \sigma^{\tilde{2}} = -\frac{1}{2}(-\sin\tilde{\psi}\, d\tilde{\theta} + \cos\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \tag{2.5}$$

$$\sigma^{\hat{3}} = -\frac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi}) \qquad \sigma^{\tilde{3}} = -\frac{1}{2}(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi}) \tag{2.6}$$

and satisfy the relations (with σ^a being either $\sigma^{\hat{a}}$ or $\sigma^{\tilde{a}}$)

$$\mathrm{d}\sigma^a = \epsilon_{abc}\sigma^b \wedge \sigma^c\,. \tag{2.7}$$

With this normalisation the metric on the unit radius round three sphere is given by

$$d\Omega_3^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\,. \tag{2.8}$$

The only non trivial field strength in our Ansatz is the Ramond-Ramond 5-form: it is more conveniently expressed in terms of the "d-bein"

$$e^0 = h^{-1}(\mathrm{d}t + V_i \mathrm{d}x^i) \tag{2.9}$$

$$e^j = h\frac{\rho_1}{\rho_3}T\delta_i^j \mathrm{d}x^i \tag{2.10}$$

$$e^3 = h\frac{\rho_1}{\rho_3}\mathrm{d}y \tag{2.11}$$

$$e^{\hat{a}} = \begin{cases} \rho_1\sigma^{\hat{a}} & \hat{a} = 1,2 \\ \rho_3(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu) & \hat{a} = 3 \end{cases} \tag{2.12}$$

$$e^{\tilde{a}} = \tilde{\rho}\sigma^{\tilde{a}} \tag{2.13}$$

as

$$F_{(5)} = 2\left(\tilde{G}_{mn}e^m \wedge e^n + \tilde{V}_m e^m \wedge e^{\hat{3}} + \tilde{g}e^{\hat{1}} \wedge e^{\hat{2}}\right) \wedge \tilde{\rho}^3 \mathrm{d}\tilde{\Omega}_3 +$$
$$2\left(-G_{pq}e^p \wedge e^q \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \star_4\tilde{V} \wedge e^{\hat{1}} \wedge e^{\hat{2}} - \star_4\tilde{g} \wedge e^{\hat{3}}\right)\,, \tag{2.14}$$

where

$$G_{mn} = \frac{1}{2}\epsilon_{mnpq}\tilde{G}^{pq} \tag{2.15}$$

$$\star_4\tilde{V} = \frac{1}{3!}\epsilon_{mnpq}\tilde{V}^m e^n \wedge e^p \wedge e^q \tag{2.16}$$

$$\star_4\tilde{g} = \tilde{g}e^0 \wedge e^1 \wedge e^2 \wedge e^3\,. \tag{2.17}$$

The complete solution can be expressed in terms of four independent functions m, n, p, T defined on the halfspace (x^1, x^2, y), as follows

$$\rho_1^4 = \frac{mp + n^2}{m}y^4 \qquad \rho_3^4 = \frac{p^2}{m(mp + n^2)} \qquad \tilde{\rho}^4 = \frac{m}{mp + n^2} \tag{2.18}$$

$$h^4 = \frac{mp^2}{mp + n^2} \qquad A_t = \frac{n - p}{p} \qquad A_i = A_t V_i - \frac{1}{2}\epsilon_{ij}\partial_j \ln T \tag{2.19}$$

and

$$
\begin{aligned}
\mathrm{d}V &= -y \star_3 [\mathrm{d}n + (nD + 2ym(n-p) + 2n/y)\mathrm{d}y] && (2.20)\\
\partial_y \ln T &= D && (2.21)\\
D &\equiv 2y(m+n-1/y^2)\,, && (2.22)
\end{aligned}
$$

where $\star_3$ indicates the Hodge dual in the three dimensional diagonal metric

$$
\mathrm{d}s_3^2 = T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2\,. \tag{2.23}
$$

The various four-dimensional forms from which the 5-form field strength is constructed are given by

$$
\begin{aligned}
\tilde{g} &= \frac{1}{4\tilde{\rho}}\left[1 - \frac{\rho_3^2}{\rho_1^2}(1+A_t)\right] && (2.24)\\
\tilde{V} &= \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) && (2.25)\\
G\rho_1^2\rho_3 &= \mathrm{d}B_t \wedge (\mathrm{d}t + V_i\mathrm{d}x^i) + B_t\mathrm{d}V + \mathrm{d}\hat{B} && (2.26)\\
\tilde{G}\tilde{\rho}^3 &= \frac{1}{2}g\rho_1^2\tilde{\rho}^3\mathrm{d}A + \mathrm{d}\tilde{B}_t \wedge (\mathrm{d}t + V_i\mathrm{d}x^i) + \tilde{B}_t\mathrm{d}V + \mathrm{d}\hat{\tilde{B}}\,, && (2.27)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{B}_t &= -\frac{1}{16}y^2\,\frac{n-1/y^2}{p}\\
\mathrm{d}\hat{\tilde{B}} &= -\frac{1}{16}y^3 \star_3 [\mathrm{d}m + 2mD\,\mathrm{d}y]\\
B_t &= -\frac{1}{16}y^2\,\frac{n}{m}\\
\mathrm{d}\hat{B} &= \frac{1}{16}y^3 \star_3 [\mathrm{d}p + 4yn(p-n)\mathrm{d}y]\,.
\end{aligned}
\tag{2.28}
$$

The Bianchi identities on $F_{(5)}$ and the integrability condition for (2.20) give three second order differential equations on m, n, p which, together with (2.21) give a system of nonlinear coupled elliptic differential equations

$$
\begin{aligned}
y^3(\partial_1^2+\partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right) + y^2\partial_y\left[T^2\left(yDn + 2y^2m(n-p)\right)\right] + 4y^2DT^2n &= 0\\
y^3(\partial_1^2+\partial_2^2)m + \partial_y\left(y^3T^2\partial_y m\right) + \partial_y\left(y^3T^2 2mD\right) &= 0\\
y^3(\partial_1^2+\partial_2^2)p + \partial_y\left(y^3T^2\partial_y p\right) + \partial_y\left[y^3T^2 4ny(n-p)\right] &= 0\\
\partial_y \ln T &= D\,.
\end{aligned}
\tag{2.29}
$$

3. $AdS_5 \times Y^{p,q}$ solutions

Taking any solution described in section 2 and assuming rotational symmetry in the $\{x^1, x^2\}$ plane, the bosonic symmetry is enhanced to $\mathbb{R} \times \mathrm{SO}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$. We will first consider a subset of solutions which preserve 8 supersymmetries (the generic solution preserves only 4 of them as explained in the previous section). The well known $AdS_5 \times Y^{p,q}$ [16] are clearly contained in this subset: the round S^3 is a factor in AdS_5, as suggested by the analysis in [12], with $\mathbb{R}\times\mathrm{SO}(4)$ the non compact version of $\mathrm{U}(1)\times\mathrm{SO}(4) \subset \mathrm{SO}(2,4)$, while the remaining $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ is the isometry group of the generic $Y^{p,q}$ metric.

3.1 Constraints for enhanced supersymmetry

Since the solutions described in [12] generically preserve only 4 supersymmetries, the $AdS_5 \times Y^{p,q}$ geometries will be specified by a set of constraints on the four functions m, n, p and T. We will now show how these constraints arise.

The supersymmetry parameters that leave invariant our background are the solutions to the Killing spinor equation

$$\delta\chi_M = \nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi = 0\,. \tag{3.1}$$

As a consequence of the symmetry assumptions we look for a solution ψ of the form

$$\psi = \varepsilon \otimes \hat{\chi} \otimes \tilde{\chi}_{(b)}\,. \tag{3.2}$$

Here ε is an 8 component complex spinor and $\hat{\chi}, \tilde{\chi}_{(b)}$ are 2 component complex spinors defined on the two 3-spheres satisfying

$$\frac{\partial}{\partial\omega^{\hat{a}}}\hat{\chi} = 0 \qquad \sigma_{\hat{3}}\hat{\chi} = s\hat{\chi} \tag{3.3}$$

$$\nabla'_{\tilde{a}}\tilde{\chi}_{(b)} = b\frac{\mathrm{i}}{2}\sigma_{\tilde{a}}\tilde{\chi}_{(b)} \tag{3.4}$$

where $\omega^{\hat{a}}, \omega^{\tilde{a}}$ are coordinates on the two spheres. ∇' is the covariant derivative on the unit radius three sphere and $s, b = \pm 1$. The spinor $\hat{\chi}$ is a singlet under the $\mathrm{SU}(2)_L$ isometry of the squashed 3-sphere, while the spinors $\tilde{\chi}_\pm$ transform as the $(0, \frac{1}{2})$ for upper sign and $(\frac{1}{2}, 0)$ for the lower sign, of the SO(4) isometry of the round S^3, which is part of AdS_5.

The analysis in [12] fixes $b = s = 1$, i.e. $\tilde{\chi}$ has definite chirality in SO(4) and $\hat{\chi}$ is highest weight of the broken $\mathrm{SU}(2)_R$. ε is proportional to some ε_0 obeying $\varepsilon_0^\dagger\varepsilon_0 = 1$. Since we have a doublet of $\tilde{\chi}_{(1)}$, the space of solutions is 2 dimensional and complex giving rise to 4 real preserved supersymmetries. We will show that $AdS_5 \times Y^{p,q}$ geometries are obtained by requiring that spinors with $b = -1$, $s = 1$ are also solutions of the equations (3.1). In this case there are two doublets of $\tilde{\chi}$ and thus 8 real solutions to (3.1). This agrees with what one expects from the $\mathcal{N} = 1$ SCFT side: there, out of the 4 pairs of Killing spinors $\xi^A_\pm$, $A = 1, \ldots, 4$ in the **4** of SU(4) of the $\mathcal{N} = 4$ theory on $\mathbb{R} \times S^3$, obeying

$$D_\mu\xi^A_\pm = \pm\frac{i}{2}\sigma_\mu\xi^A_\pm \tag{3.5}$$

only the SU(3) singlet $\xi_\pm$ in $\mathrm{SU}(3) \times \mathrm{U}(1) \subset \mathrm{SU}(4)$ survives in the $\mathcal{N} = 1$ case. This has SU(4) weights $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and, picking the SO(4) inside SU(4) corresponding for example to the first two entries, we see that it is a singlet of, say, $\mathrm{SU}(2)_L$ and highest weight of $\mathrm{SU}(2)_R$ in the $\mathrm{SO}(4) \subset \mathrm{SU}(4)$. Furthermore, the two signs in (3.5) correspond to the two chiralities of the SO(4) isometry group of S^3. Since this S^3 corresponds to the S^3 inside AdS_5, this checks with the above requirement of $b = \pm 1$.

Due to the conditions on the spinor, $\hat{\chi}$ and $\tilde{\chi}_{(b)}$ factorise in each component of the gravitino variation equation which then becomes equivalent to the following system of coupled differential and algebraic equations on ε[1]

$$\left[\tilde{\nabla}_\mu - \frac{1}{4}F_{\mu\nu}\Xi^{\nu}{}_{m}\gamma^m\gamma^5\hat{\sigma}_1 s + \mathrm{i}A_\mu s - \left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\gamma_\mu\right]\varepsilon = 0 \tag{3.7}$$

$$\left[\frac{\mathrm{i}}{2}\frac{\rho_3}{\rho_1}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\not{\partial}\rho_1 + \rho_1\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s - \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{3.8}$$

$$\left[\frac{\mathrm{i}}{2}\left(2 - \frac{\rho_3^2}{\rho_1^2}\right)\gamma_5\hat{\sigma}_1 + \frac{1}{2}\not{\partial}\rho_3 + \frac{1}{8}\rho_3^2\not{F}\gamma_5\hat{\sigma}_1 s + \rho_3\left(\tilde{\not{G}} - \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{3.9}$$

$$\left[\frac{\mathrm{i}}{2}b\gamma_5\hat{\sigma}_2 + \frac{1}{2}\not{\partial}\tilde{\rho} - \tilde{\rho}\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0. \tag{3.10}$$

Note that the first equation is a first order differential 4-vector equation for ε while the last three are algebraic 4-scalar equations.

We now express all the supergravity fields via the functions m, n, p and T of the previous section. We are thus guaranteed that a solution to the above system with $b = s = 1$ exists by the analysis in [12]. We now ask that a second solution to these equations exists for $b = -1$, $s = 1$.

We have used Mathematica to solve explicitly the equations. The existence of solutions implies certain constraints on the background, which are more conveniently expressed in terms of the metric entries as

$$\begin{aligned} 1 + A_t &= \rho_1^2/\rho_3^2 \\ \rho_1^2 - \rho_3^2 &= S^4/\rho_1^2 \\ T^2\partial_y \ln(\rho_1/\rho_3)\left[2\rho_1^2/\rho_3^2 - 2 + y\partial_y \ln(\rho_1/\rho_3)\right] + y\left[\partial_r \ln(\rho_1/\rho_3)\right]^2 &= 0 \end{aligned} \tag{3.11}$$

where (r, ϕ) are polar coordinates in the (x^1, x^2) plane. Notice that the first two constraints together with the relation $y = \rho_1\tilde{\rho}$ allow us to express the four functions $\rho_1, \rho_3, \tilde{\rho}, A_t$ in terms of only one function. The last constraint together with the equation for $\partial_y T$ can be used to eliminate T. Moreover, the three second order differential equations that came from the integrability condition for the 1/8 supersymmetric geometries are reduced to a single equation which is more easily expressed in terms of the function $\tilde{z}$

$$\tilde{z} \equiv \frac{1}{2}\left[1 + \tanh\left(\frac{\rho_3(1 + A_t)}{\tilde{\rho}}\right)\right] \tag{3.12}$$

$$\frac{1}{r}\partial_r\left(r\partial r\tilde{z}\right) + y\partial_y\left\{T^2\frac{1}{y}\left[\partial_y\tilde{z} + 4\tilde{z}(1 - \tilde{z})\frac{\rho_1^2/\rho_3^2 - 1}{y}\right]\right\} = 0 \tag{3.13}$$

[1]For example the first equation is obtained as follows

$$\begin{aligned} (\nabla_\mu + M\Gamma_\mu)\psi &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^{\nu}{}_{m}\Gamma^m\Gamma^{\hat{3}} + A_\mu\left(\Sigma_{\hat{3}} + \Gamma^{\hat{1}}\Gamma^{\hat{2}}\right) - A_\mu\nabla_{\hat{3}} + M\Gamma_\mu\right)\psi = \\ &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^{\nu}{}_{m}\Gamma^m\Gamma^{\hat{3}} + A_\mu\Gamma^{\hat{1}}\Gamma^{\hat{2}} + M\left(\Gamma_\mu + A_\mu\rho_3\Gamma_{\hat{3}}\right)\right)\psi = \\ &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^{\nu}{}_{m}\gamma^m\sigma^{\hat{3}} + A_\mu\sigma_{\hat{3}} + M\gamma_\mu\right)\psi \end{aligned} \tag{3.6}$$

where the combination ρ_1^2/ρ_3^2 is given by

$$\frac{\rho_1^2}{\rho_3^2} = \left(\frac{1 + \sqrt{1 - 4S^4 \frac{1-\tilde{z}}{y^2 \tilde{z}}}}{2} \right)^{-1} \tag{3.14}$$

and T can be found by solving the third equation in (3.11). The solution is thus specified completely by a single function.[2]

3.2 $AdS_5 \times Y^{p,q}$ metrics

In this section, we are going to show how the $AdS_5 \times Y^{p,q}$ geometries arise from the generic description given above. As a first step we present the conditions that should be satisfied by the 1/4 supersymmetric solutions in order that they factorise into,

$$AdS_5 \times X^5\,, \tag{3.15}$$

For some supersymmetric five-manifold X^5. These will turn out to be equivalent to a single first order differential equation which implies the second order equation in (3.13). The opposite in general cannot be proven: the generic solution preserving 8 supersymmetries apparently is not factorisable in general.

As a second step we will prove, by giving the explicit coordinate transformation to the gauge in [16], that the X^5 factor is indeed a generic $Y^{p,q}$ manifold.

First of all we switch to the more convenient coordinates $(\tilde{\rho}, \rho_1, \tilde{\phi}, \hat{\psi}')$ defined by

$$\begin{aligned} y &= \rho_1 \tilde{\rho} \\ r &= r(\rho_1, \tilde{\rho}) \\ \phi &= \tilde{\phi} + \tilde{c}\, t \\ \hat{\psi} &= \hat{\psi}' - 2\gamma\, t - 2\delta\, \phi \equiv \hat{\psi}' - (2\gamma + 2\tilde{c}\,\delta)t - 2\delta\, \tilde{\phi} \end{aligned} \tag{3.16}$$

Using the constraints in (3.11) the solution is completely specified once the explicit form of the function $r(\rho_1, \tilde{\rho})$ is known.

The last shift implies that the left invariant one-form $\sigma^{\hat{3}}$ is shifted to $\sigma^{\hat{3}\,\prime} + (\gamma + \tilde{c}\,\delta)\mathrm{d}t + \delta\, \mathrm{d}\tilde{\phi}$. With a slight abuse of notation we will keep calling this shifted one form $\sigma^{\hat{3}}$. The metric of 2.1 is thus

$$\begin{aligned} \mathrm{d}s^2 &= -h^{-2}(\mathrm{d}t^2 + V_\phi \mathrm{d}\phi)^2 + h^2 \frac{\rho_1^2}{\rho_3^2}(T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2) + \\ &\quad + \tilde{\rho}^2 \mathrm{d}\Omega_3^2 + \rho_1^2 \big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2 \big] + \rho_3^2 (\sigma^{\hat{3}} - A_t \mathrm{d}t - A_\phi \mathrm{d}\phi)^2 \\ &= g_{tt} \mathrm{d}t^2 + g_{\tilde{\rho}\tilde{\rho}} \mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2 + 2 g_{t\tilde{\phi}} \mathrm{d}t \mathrm{d}\tilde{\phi} + 2 g_{\rho_1 \tilde{\rho}} \mathrm{d}\rho_1 \mathrm{d}\tilde{\rho} + \\ &\quad + g_{\rho_1 \rho_1} \mathrm{d}\rho_1^2 + g_{\tilde{\phi}\tilde{\phi}} \mathrm{d}\tilde{\phi}^2 + \rho_1^2 \big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2 \big] + \\ &\quad + \rho_3^2 \Big[\sigma^{\hat{3}} + \big(\gamma - A_t - \tilde{c}(A_\phi - \delta)\big) \mathrm{d}t - (A_\phi - \delta) \mathrm{d}\tilde{\phi} \Big]^2 , \end{aligned} \tag{3.17}$$

[2] If, instead of doubling supersymmetry by requiring $b = -1$ and $s = 1$ in addition to $b = s = 1$, one requires to have solutions of (3.1) also for $b = s = -1$, then one obtains a different set of constraints on the background. By making an asymptotic analysis similar to the one we will perform here in section 4, it can be shown that the resulting geometry describes 1/4-BPS states in the background $AdS_5 \times S^5$

with

$$
\begin{aligned}
g_{tt} &= -h^{-2}(1+\tilde{c}V_\phi)^2 + \tilde{c}^2 h^2 \frac{\rho_1^2}{\rho_3^2} T^2 r^2 \\
g_{\tilde{\rho}\tilde{\rho}} &= h^2 \frac{\rho_1^2}{\rho_3^2}\left[\rho_1^2 + T^2 r^2 \left(\frac{\partial \ln r}{\partial \tilde{\rho}}\right)^2\right] \\
g_{t\tilde{\phi}} &= -h^{-2}(1+\tilde{c}V_\phi)V_\phi + \tilde{c} h^2 \frac{\rho_1^2}{\rho_3^2} T^2 r^2 \\
g_{\rho_1\tilde{\rho}} &= h^2 \frac{\rho_1^2}{\rho_3^2}\left[\rho_1\tilde{\rho} + T^2 r^2 \frac{\partial \ln r}{\partial \tilde{\rho}}\frac{\partial \ln r}{\partial \rho_1}\right] \\
g_{\rho_1\rho_1} &= h^2 \frac{\rho_1^2}{\rho_3^2}\left[\tilde{\rho}^2 + T^2 r^2 \left(\frac{\partial \ln r}{\partial \rho_1}\right)^2\right] \\
g_{\tilde{\phi}\tilde{\phi}} &= -h^{-2}V_\phi^2 + h^2 \frac{\rho_1^2}{\rho_3^2} T^2 r^2
\end{aligned}
\tag{3.18}
$$

We recall the constraint on the metric components coming from the requirement of 1/4 supersymmetry,

$$
\begin{aligned}
1 + A_t &= \frac{\rho_1^2}{\rho_3^2} \\
\rho_1^2 - \rho_3^2 &= \frac{S^4}{\rho_1^2} \\
h^{-2} &= \tilde{\rho}^2 + \rho_1^4/\rho_3^2.
\end{aligned}
\tag{3.19}
$$

In order that the metric factorises we need the $\mathrm{d}t\,\sigma^{\hat{3}}$ term to vanish which requires that

$$
A_t + \tilde{c}(A_\phi - \delta) = \gamma\,. \tag{3.20}
$$

Imposing also $g_{t\tilde{\phi}} = 0$ we obtain

$$
\tilde{c} h^2 \frac{\rho_1^2}{\rho_3^2} T^2 r^2 = h^{-2}(1+\tilde{c}V_\phi)V_\phi\,. \tag{3.21}
$$

In order to have an AdS_5 factor we should have $-g_{tt} = L^2 + \tilde{\rho}^2$ which gives, using the last relation

$$
h^{-2}(1+\tilde{c}V_\phi) = L^2 + \tilde{\rho}^2\,, \tag{3.22}
$$

We also demand that $g_{\tilde{\rho}\tilde{\rho}} = \frac{L^2}{L^2+\tilde{\rho}^2}$ which after a little bit of algebra gives

$$
\frac{\partial \ln r}{\partial \tilde{\rho}} = \pm \frac{\tilde{c}\tilde{\rho}}{L^2 + \tilde{\rho}^2}. \tag{3.23}
$$

Requiring that we have a product metric means that we also must impose that $g_{\rho_1\tilde{\rho}} = 0$ which implies

$$
\frac{\partial \ln r}{\partial \rho_1} = \mp \tilde{c} \frac{\rho_1^3}{\rho_3^2 L^2 - \rho_1^4}. \tag{3.24}
$$

JHEP02(2008)050

As a result we find immediately that

$$g_{\rho_1\rho_1} = \frac{\rho_1^2 L^2}{\rho_3^2 L^2 - \rho_1^4} \tag{3.25}$$

and

$$g_{\tilde{\phi}\tilde{\phi}} = \frac{1}{\tilde{c}^2}\left(L^2 - \frac{\rho_1^4}{\rho_3^2}\right) . \tag{3.26}$$

The generic solutions to equation (3.23) for the upper sign are

$$r = (L^2 + \tilde{\rho}^2)^{\tilde{c}/2} r_0(\rho_1)\rho_1^{\tilde{c}} \tag{3.27}$$

where we have extracted the $\rho_1^{\tilde{c}}$ for future convenience. (3.24) is an equation for $r_0(\rho_1)$

$$r_0'(\rho_1) = \tilde{c}\frac{L^2(\rho_1^4 - S^4)}{\rho_1^7 - L^2\rho_1(\rho_1^4 - S^4)} r_0(\rho_1) \tag{3.28}$$

Using the last constraint in (3.11)

$$T^2 \partial_y \ln(\rho_1/\rho_3)\big[2\rho_1^2/\rho_3^2 - 2 + y\partial_y \ln(\rho_1/\rho_3)\big] + y\big[\partial_r \ln(\rho_1/\rho_3)\big]^2 = 0 \tag{3.29}$$

we can find T. Note that both the first order differential equation for T

$$\partial_y \ln T = D \tag{3.30}$$

and the second order equation in (3.13) are satisfied when $r_0(\rho_1)$ satisfies the equation (3.28).

3.3 Relation to standard $Y^{p,q}$ coordinates

Now we show the coordinates transformation that brings the metric on X_5 to the standard metric on $Y^{p,q}$ as presented in [16]. We perform the rescaling

$$\tilde{\rho} \to L\tilde{\rho}, \qquad \rho_i \to L\rho_i, \qquad S \to LS \tag{3.31}$$

which takes the metric of AdS_5 into the form

$$\mathrm{d}s^2_{AdS_5} = L^2\left(-(\tilde{\rho}^2+1)\mathrm{d}t^2 + \frac{\mathrm{d}\tilde{\rho}^2}{\tilde{\rho}^2+1} + \tilde{\rho}^2\mathrm{d}\Omega_3^2\right) \tag{3.32}$$

while the metric on the "internal" part is

$$\begin{aligned}\mathrm{d}s_5^2 = L^2\bigg[&\frac{\rho_1^2}{\rho_3^2 - \rho_1^4}\mathrm{d}\rho_1^2 + \frac{1}{\tilde{c}^2}\left(1 - \frac{\rho_1^4}{\rho_3^2}\right)\mathrm{d}\tilde{\phi}^2 + \\ &+ \rho_1^2\big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\big] + \rho_3^2\big(\sigma^{\hat{3}} - (A_\phi - \delta)\mathrm{d}\tilde{\phi}\big)^2\bigg] .\end{aligned} \tag{3.33}$$

The standard form for the metric on $Y^{p,q}$ [16] is,

$$\begin{aligned}\mathrm{d}s^2 = &\frac{1 - c\hat{y}}{6}(\mathrm{d}\hat{\theta}^2 + \sin^2\hat{\theta}\mathrm{d}\hat{\phi}^2) + \frac{1}{w(\hat{y})q(\hat{y})}d\hat{y}^2 + \frac{q(\hat{y})}{9}(\mathrm{d}\hat{\psi} + \cos\hat{\theta}\mathrm{d}\hat{\phi})^2 \\ &+ w(\hat{y})\left[\mathrm{d}\alpha - \frac{ac - 2\hat{y} + \hat{y}^2 c}{6(a - \hat{y}^2)}(\mathrm{d}\hat{\psi} + \cos\hat{\theta}\mathrm{d}\hat{\phi})\right]^2\end{aligned} \tag{3.34}$$

with[3]

$$
\begin{aligned}
w(\hat{y}) &= \frac{2(a-\hat{y}^2)}{1-c\hat{y}} \\
q(\hat{y}) &= \frac{a-3\hat{y}^2+2c\hat{y}^3}{a-\hat{y}^2}\ .
\end{aligned}
\tag{3.35}
$$

Recalling that

$$
\begin{aligned}
\sigma^{\hat{1}} &= -\frac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\ \sin\hat{\theta}\, d\hat{\phi}) \\
\sigma^{\hat{2}} &= -\frac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\ \sin\hat{\theta}\, d\hat{\phi}) \\
\sigma^{\hat{3}} &= -\frac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi})
\end{aligned}
\tag{3.36}
$$

we immediately get

$$
\rho_1^2 = \frac{2}{3}(1-c\hat{y})
\tag{3.37}
$$

and

$$
\frac{1}{4}\rho_3^2 = \frac{2+ac^2-6c\hat{y}+3c^2\hat{y}^2}{18(1-c\hat{y})}\,.
\tag{3.38}
$$

Recalling that $\rho_1^2-\rho_3^2 = S^4/\rho_1^2$ we have

$$
S^4 = \frac{4}{27}(1-ac^2)\,.
\tag{3.39}
$$

We also have

$$
A_\phi = \frac{1}{\tilde{c}}(\gamma - A_t) + \delta = \frac{1}{\tilde{c}}\left(\gamma + 1 - \frac{\rho_1^2}{\rho_3^2}\right) + \delta\,.
\tag{3.40}
$$

Assuming that $\alpha = \beta\tilde{\phi}$ and equating the $\mathrm{d}\tilde{\phi}^2$ component of the metric we get

$$
\frac{1}{\tilde{c}^2}\left(1-\frac{\rho_1^4}{\rho_3^2}\right) + \rho_3^2(A_\phi-\delta)^2 = w(\hat{y})\beta^2
\tag{3.41}
$$

which implies after some straightforward algebra that

$$
\gamma = \frac{1}{2} \quad , \quad \beta = \pm\frac{c}{2\tilde{c}}\,.
\tag{3.42}
$$

The coefficient of the cross term $\mathrm{d}\tilde{\phi}\,\sigma^{\hat{3}}$ is

$$
\frac{1}{2}\rho_3^2(A_\phi-\delta) = -\beta w(\hat{y})\frac{ac-2\hat{y}+\hat{y}^2c}{6(a-\hat{y}^2)}
\tag{3.43}
$$

which implies that we must have $\beta = -\frac{c}{2\tilde{c}}$. We can therefore set

$$
\beta = -1 \qquad \tilde{c} = \frac{1}{2}c\,.
\tag{3.44}
$$

[3]Notice that the $\hat{\psi}$ of [16] has the opposite sign to that used here.

Using the expression for r

$$
\begin{aligned}
r &= (L^2+\tilde{\rho}^2)^{\tilde{c}/2} r_0(\rho_1)\rho_1^{\tilde{c}} \\
r_0'(\rho_1) &= \tilde{c}\frac{L^2(\rho_1^4-S^4)}{\rho_1^7-L^2\rho_1(\rho_1^4-S^4)} r_0(\rho_1)
\end{aligned}
\tag{3.45}
$$

and the definition

$$
A_\phi = A_t V_\phi + \frac{1}{2} r\partial_r \ln T \tag{3.46}
$$

we get

$$
A_t + \tilde{c}(A_\phi - \delta) = \frac{1}{2} - \frac{\tilde{c}}{2}(1+2\delta) \tag{3.47}
$$

which gives

$$
\delta = -\frac{1}{2} \tag{3.48}
$$

for $c, \tilde{c} \neq 0$. Finally, the matching of the $\mathrm{d}\hat{y}^2$ factor

$$
\frac{1}{9}\frac{c^2}{\rho_3^2-\rho_1^4} = \frac{1}{w(\hat{y})q(\hat{y})} \tag{3.49}
$$

is identically satisfied. Finally, we observe that the polynomial $q(\hat{y}) = a - 3\hat{y}^2 + 2c\hat{y}^3$, whose zeroes $\hat{y}_1$ and $\hat{y}_2$, with $\hat{y}_1 < 0$ and $\hat{y}_2$ the smallest between the two other positive zeroes, determine the range of $\hat{y}$, $\hat{y}_1 \leq \hat{y} \leq \hat{y}_2$, can be expressed in terms of ρ_1 as $q = -27(\rho_1^6 - \rho_1^4 + S^4)/4$. Notice also that in the metric (3.34) any non zero value of c can be reabsorbed in a rescaling of $\hat{y}$ and α. We may thus set $c = 1$ whenever $c \neq 0$.

c=0 case. Let us now take a look at the singular case

$$
c = \tilde{c} = 0 \tag{3.50}
$$

which corresponds to the Sasaki-Einstein internal manifold $Y^{1,0} \equiv T^{1,1}$. From the equations (3.21),(3.47) we can immediately obtain

$$
\begin{aligned}
A_t &= \gamma = \frac{1}{2} \\
V_\phi &= 0
\end{aligned}
\tag{3.51}
$$

and from (3.37),(3.38),(3.39)

$$
\begin{aligned}
\rho_1^2 &= \rho_3 = \frac{2}{3} \\
S &= \frac{4}{27}\,.
\end{aligned}
\tag{3.52}
$$

Notice that the equation for ρ_1 implies

$$
y = \sqrt{\frac{2}{3}} L^2 \tilde{\rho} \tag{3.53}
$$

Given these explicit values for ρ_1 and ρ_3, the last constraint in (3.11)

$$
T^2\partial_y \ln(\rho_1/\rho_3)\big[2\rho_1^2/\rho_3^2 - 2 + y\partial_y \ln(\rho_1/\rho_3)\big] + y\big[\partial_r \ln(\rho_1/\rho_3)\big]^2 = 0 \tag{3.54}
$$

is automatically satisfied.

The metric on the internal manifold becomes

$$\mathrm{d}s_5^2 = L^2\left[\frac{3}{2}\tau(r)^2\left(\frac{\mathrm{d}r^2}{r^2}+\mathrm{d}\tilde{\phi}^2\right)+\frac{2}{3}\left[\left(\sigma^{\hat{1}}\right)^2+\left(\sigma^{\hat{2}}\right)^2\right]+\frac{4}{9}\left(\sigma^{\hat{3}}-(A_\phi-\delta)\mathrm{d}\tilde{\phi}\right)^2\right] \tag{3.55}$$

where

$$T^2 = (\tilde{\rho}^2+1)\frac{\tau(r)^2}{r^2} \tag{3.56}$$

is such that T solves the equation

$$\partial_y \ln T = D \iff \partial_{\tilde{\rho}} \ln T = \frac{2\tilde{\rho}}{\tilde{\rho}^2+1} \tag{3.57}$$

We now match this expression with the one in [16]. For $c=0$, a can be reabsorbed in a coordinate redefinition. We set, for convenience,

$$a = 3 \tag{3.58}$$

and obtain,

$$\begin{aligned}\mathrm{d}s^2 = {} & \frac{1}{6}(\mathrm{d}\hat{\theta}^2+\sin^2\hat{\theta}\mathrm{d}\hat{\phi}^2)+\frac{1}{6(1-\hat{y}^2)}\mathrm{d}\hat{y}^2+\frac{1-\hat{y}^2}{3(3-\hat{y}^2)}(\mathrm{d}\hat{\psi}+\cos\hat{\theta}\mathrm{d}\hat{\phi})^2\\ & +2(3-\hat{y}^2)\left[\mathrm{d}\alpha+\frac{2\hat{y}}{6(3-\hat{y}^2)}(\mathrm{d}\hat{\psi}+\cos\hat{\theta}\mathrm{d}\hat{\phi})\right]^2\end{aligned} \tag{3.59}$$

Assuming, as in the generic case, $\alpha \equiv -\tilde{\phi}$, and equating the $g_{3\alpha}$ and $g_{\alpha\alpha}$ components we get

$$A_\phi - \delta = -3\hat{y} \tag{3.60}$$

and

$$\frac{3}{2}\tau^2 + 4\hat{y}^2 = 2(3-\hat{y}^2) \quad \Rightarrow \quad \tau^2 = 4(1-\hat{y}^2) \tag{3.61}$$

Assuming $r = r(\hat{y})$ and equating the $\mathrm{d}\hat{y}^2$ term gives

$$\frac{\partial \ln r}{\partial \hat{y}} = \pm\frac{1}{6(1-\hat{y}^2)} \quad \Rightarrow \quad r = \lambda\left(\frac{1-\hat{y}}{1+\hat{y}}\right)^{\mp 1/12} \tag{3.62}$$

where λ is an arbitrary constant and we fix $\lambda = 1$. We are now able to determine the constant δ through the equation

$$A_\phi = A_t V_\phi + \frac{1}{2} r\partial_r \ln T = -\frac{1}{2} \mp 3\hat{y} \tag{3.63}$$

which fixes the upper choice for the sign and

$$\delta = -\frac{1}{2} \tag{3.64}$$

In order to bring the metric to the standard $T^{1,1}$ form we set

$$\hat{y} = -\cos\tilde{\theta} \tag{3.65}$$

which gives

$$r = \left(\tan\frac{\tilde\theta}{2}\right)^{1/6} \quad , \quad \tau = 4\sin^2\tilde\theta \tag{3.66}$$

and thus

$$\frac{\mathrm{d}s_5^2}{L^2} = \frac{1}{6}\left(\mathrm{d}\tilde\theta^2 + 36\sin^2\tilde\theta\mathrm{d}\tilde\phi^2\right) + \frac{1}{6}\left(\mathrm{d}\hat\theta^2 + \sin\hat\theta\mathrm{d}\hat\phi^2\right) + \frac{1}{9}\left(\mathrm{d}\hat\psi + \cos\hat\theta\mathrm{d}\hat\phi + 6\cos\tilde\theta\mathrm{d}\tilde\phi\right)^2 \tag{3.67}$$

which is the $T^{1,1}$ metric up to the trivial rescaling

$$\tilde\phi \to \frac{1}{6}\tilde\phi \tag{3.68}$$

4. Asymptotic expansion for half BPS states in $AdS_5 \times Y^{p,q}$

In this section we study generic asymptotic perturbations of the $AdS_5 \times Y^{p,q}$ geometries that preserve 1/2 of the bulk supersymmetries. We relax the constraints of (3.11) which give back $AdS_5 \times Y^{p,q}$ and solve the differential equations (2.29) with the boundary conditions that the solutions approach $AdS_5 \times Y^{p,q}$ at large distances (including also the particular case $c = 0$). We will work in the somewhat mixed coordinates $(y, \hat y)$ or $(y, \tilde\theta)$ and solve the equation in an expansion for large y, with the simplifying assumption that the solutions are invariant under shifts in $\tilde\phi$. We make the following Ansatz for the expansion of our functions,

$$\rho_1 = L\sqrt{\frac{2}{3}(1-c\hat y)}\left(1 + \rho_1^{(1)}(\hat y)\frac{L^4}{y^2} + \rho_1^{(2)}(\hat y)\frac{L^8}{y^4}\right) \tag{4.1}$$

$$\rho_3 = L\sqrt{\frac{2(2+ac^2-6c\hat y+3c^2\hat y^2)}{9(1-c\hat y)}}\left(1 + \rho_3^{(1)}(\hat y)\frac{L^4}{y^2} + \rho_3^{(2)}(\hat y)\frac{L^8}{y^4}\right) \tag{4.2}$$

$$\tilde\rho = \frac{y}{\rho_1} \tag{4.3}$$

$$A_t = \frac{1-ac^2}{2+ac^2-6c\hat y+3c^2\hat y^2}\left(1 + A_t^{(1)}(\hat y)\frac{L^4}{y^2} + A_t^{(2)}(\hat y)\frac{L^8}{y^4}\right) \tag{4.4}$$

$$T = \frac{y}{r}\sqrt{\frac{2(a-3\hat y^2+2c\hat y^3)}{(1-c\hat y)^3}}\left(1 + t^{(1)}(\hat y)\frac{L^4}{y^2} + t^{(2)}(\hat y)\frac{L^8}{y^4}\right) \tag{4.5}$$

$$V_\phi = \frac{4c(1-c\hat y)(a-3\hat y^2+2c\hat y^3)}{3(2+ac^2-6c\hat y+3c^2\hat y^2)}\frac{L^4}{y^2} + V_\phi^{(2)}(\hat y)\frac{L^8}{y^4} + V_\phi^{(3)}(\hat y)\frac{L^{12}}{y^6} \tag{4.6}$$

$$r = y^{c/2}r_0(\hat y) \tag{4.7}$$

This expansion reproduces the $c = 0$ limit upon setting $a = 3$, as in the previous section.

In these coordinates, the condition (3.28) becomes

$$r_0'(\hat y) = \frac{2+ac^2-6c\hat y+3c^2\hat y^2}{4(1-c\hat y)(a-3\hat y^2+2c\hat y^3)}r_0(\hat y) \tag{4.8}$$

The functions m, n, p are given by

$$m = \frac{1}{\rho_3^2[\tilde{\rho}^2 + (1+A_t)^2\rho_3^2]} \tag{4.9}$$

$$n = \frac{(1+A_t)\rho_1^2}{y^2[\tilde{\rho}^2 + (1+A_t)^2\rho_3^2]} \tag{4.10}$$

$$p = \frac{\rho_1^2}{y^2[\tilde{\rho}^2 + (1+A_t)^2\rho_3^2]} \,. \tag{4.11}$$

The constraints in (3.11) and the equations (3.12) are satisfied at leading order in y. We rewrite the generic equations (2.29) in polar coordinates dividing them by T^2 and exploiting the $\mathrm{U}(1)$ symmetry of our solutions

$$\begin{aligned}
\frac{y^3}{r^2T^2} r\partial_r(r\partial_r n) + \partial_y\left(y^3\partial_y n\right) + y^2\partial_y\left[\left(yDn + 2y^2 m(n-p)\right)\right]& \\
+2y^2D(2n + y\partial_y n + yDn) &= 0 \\
\frac{y^3}{r^2T^2} r\partial_r(r\partial_r m) + \partial_y\left(y^3\partial_y m\right) + \partial_y\left(y^3 2mD\right) + 2Dy^3(\partial_y m + 2Dm) &= 0 \\
\frac{y^3}{r^2T^2} r\partial_r(r\partial_r p) + \partial_y\left(y^3\partial_y p\right) + \partial_y\left[y^3 4ny(n-p)\right] + 2Dy^3[\partial_y p + 4ny(n-p)] &= 0 \\
\partial_y \ln T &= D \,.
\end{aligned} \tag{4.12}$$

where

$$\partial_y f(y,\hat{y}) \equiv \left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_r = -\frac{c\, r_0(\hat{y})}{2r_0'(\hat{y})} \left.\frac{\mathrm{d}f}{\mathrm{d}\hat{y}}\right|_y + \left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{\hat{y}} \tag{4.13}$$

$$r\partial_r f(y,\hat{y}) \equiv r \left.\frac{\mathrm{d}f}{\mathrm{d}r}\right|_y = \frac{r_0(\hat{y})}{r_0'(\hat{y})} \left.\frac{\mathrm{d}f}{\mathrm{d}\hat{y}}\right|_y \tag{4.14}$$

The generic asymptotic solutions to these equation are specified, at each order, by 7 integration constants. As in [12], requiring regularity of the solutions implies that not all of them are independent and indeed we have only three independent integration constants.

For the case of $T^{1,1}$ asymptotics, specified by $c = 0$ the first subleading corrections are given by:

$$\rho_1^{(1)}(\tilde{\theta}) = -k + C_1 \cos\tilde{\theta} \tag{4.15}$$

$$\rho_3^{(1)}(\tilde{\theta}) = \rho_1^{(1)}(\tilde{\theta}) + k^{(1)}(\tilde{\theta}) \tag{4.16}$$

$$k^{(1)}(\tilde{\theta}) = k \tag{4.17}$$

$$A_t^{(1)}(\tilde{\theta}) = C_2 - 4C_1 \cos\tilde{\theta} \tag{4.18}$$

$$t^{(1)}(\tilde{\theta}) = \frac{L^2\sqrt{2/3}(1+9k)\sin\tilde{\theta}}{\tan\frac{\tilde{\theta}}{2}} \tag{4.19}$$

$$V_\phi^{(2)}(\tilde{\theta}) = -\frac{8}{3} C_2 \sin^2\tilde{\theta} \tag{4.20}$$

while in the generic case we get:

$$\begin{aligned}
\rho_1^{(1)}(\hat{y}) =& \frac{A[2c^2K + 9Ak + 4\hat{y}BC_1]}{6(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)}\\
\rho_3^{(1)}(\hat{y}) =& \rho_1^{(1)}(\hat{y}) + k^{(1)}(\hat{y})\\
k^{(1)}(\hat{y}) =& \frac{A[4c^2LK + 9\big(-2+8c\hat{y}-3c^3\hat{y}^3-ac^2(4-c\hat{y})\big)k]}{6(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)}\\
A_t^{(1)}(\hat{y}) =& \frac{-4c^2A^3K}{(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2} + \frac{2A}{L}C_2 - \frac{4}{3}AC_1 +\\
& - \frac{9c^2\ A[a^2c^2+\hat{y}^2(12-26c\hat{y}+21c^2\hat{y}^2-6c^3\hat{y}^3)+2a(-2+3c\hat{y}-3c^2\hat{y}^2+c^3\hat{y}^3)]}{2LB}k\\
t^{(1)}(\hat{y}) =& \frac{A(4L-27k)}{6B}
\end{aligned}$$

where

$$\begin{aligned}
A &= 1 - c\hat{y}\\
B &= 2 + ac^2 - 6c\hat{y}^2 + 2c\hat{y}^3\\
K &= a - 3\hat{y}^2 + 2c\hat{y}^3\\
L &= 1 - ac^2
\end{aligned}$$

The three arbitrary integration constants, C_1, C_2, k will turn out to be related to the supergravity dual of the flavour and baryon charge of the solutions. The second order regular solutions are rather complicated. In general, they will involve new integrations constants together with a inhomogeneous part. The expressions for the inhomogeneous part can be found in the appendix.

As already noticed, any $c \neq 0$ can be reabsorbed by a redefinition of $\hat{y}$ and so we set $c = 1$.

5. U(1) charges

We will now show how the first subleading corrections described in the previous section give rise to the Kaluza-Klein reduction of type IIB supergravity on the $Y^{p,q}$ manifolds respecting the symmetry of our Ansatz. We will calculate the global charges of the solutions under three U(1) massless KK gauge fields living in AdS_5; two of them can be identified with the KK modes of the metric associated to the Killing vectors ∂_α and $\partial_{\hat{\psi}}$ and which are dual to the flavour and R charges of the dual quiver gauge theory (more precisely to linear combinations of the charges) while the third one is associated to the expansion of the RR 4-form potential on the cohomology of $Y^{p,q}$ and it is dual to the baryon charge of the gauge theory. Since the third Betti number of such manifolds is one there is only one baryon charge.

In general, the metric on the compact manifold is modified by the metric KK gauge fields as

$$\mathrm{d}s^2 = g_{\alpha\beta}(\mathrm{d}\xi^\alpha + K_I^\alpha A_\mu^I \mathrm{d}x^\mu)(\mathrm{d}\xi^\beta + K_I^\beta A_\mu^I \mathrm{d}x^\mu) \tag{5.1}$$

where ξ^α are coordinates in $Y^{p,q}$ and x^μ in AdS_5 and

$$K_I = K_I^\alpha \partial_\alpha \qquad I = 1, \dots, n \tag{5.2}$$

are n Killing vectors of $Y^{p,q}$. In our case, only two gauge fields are turned on and they are associated to ∂_α and $\partial_{\hat\psi}$. We denote the two global gauge charges respectively as J and Q. The leading order of the corresponding gauge fields A_J, A_Q is thus given by

$$A_J = \frac{J}{\tilde\rho^2}\mathrm{d}t \qquad A_Q = \frac{Q}{\tilde\rho^2}\mathrm{d}t\,. \tag{5.3}$$

The metric is modified by the shifts

$$\mathrm{d}\hat\psi \to \mathrm{d}\hat\psi + \frac{Q}{\tilde\rho^2}\mathrm{d}t \tag{5.4}$$

$$\mathrm{d}\alpha \to \mathrm{d}\alpha + \frac{J}{\tilde\rho^2}\mathrm{d}t \tag{5.5}$$

to

$$\begin{aligned} \mathrm{d}s^2 L^{-2} = \frac{1-c\hat y}{6}(\mathrm{d}\hat\theta^2 + \sin^2\hat\theta \mathrm{d}\hat\phi^2) + \frac{1}{w(\hat y)q(\hat y)}d\hat y^2 + \frac{q(\hat y)}{9}\left(\mathrm{d}\hat\psi + \frac{Q}{\tilde\rho^2}\mathrm{d}t + \cos\hat\theta\mathrm{d}\hat\phi\right)^2 \\ + w(\hat y)\left[\mathrm{d}\alpha + \frac{J}{\tilde\rho^2}\mathrm{d}t - \frac{ac - 2\hat y + \hat y^2 c}{6(a-\hat y^2)}\left(\mathrm{d}\hat\psi + \frac{Q}{\tilde\rho^2}\mathrm{d}t + \cos\hat\theta\mathrm{d}\hat\phi\right)\right]^2 . \end{aligned} \tag{5.6}$$

Given the expression above for the metric and the solution of the equations of motion up to the first sub-leading order we obtain

$$Q = 3C_2 - 2C_1, \tag{5.7}$$

$$J = \frac{1}{2}C_2 - C_1. \tag{5.8}$$

Similarly, in the case of $T^{1,1}$ we have

$$\begin{aligned} \mathrm{d}s^2 L^{-2} = \frac{1}{6}\left(\mathrm{d}\tilde\theta^2 + 36\sin^2\tilde\theta\left(\mathrm{d}\tilde\phi + \frac{J}{\tilde\rho^2}\mathrm{d}t\right)^2\right) + \frac{1}{6}(\mathrm{d}\hat\theta^2 + \sin\hat\theta\mathrm{d}\hat\phi^2) + \\ + \frac{1}{9}\left(\mathrm{d}\hat\psi + \frac{Q}{\tilde\rho^2}\mathrm{d}t + \cos\hat\theta\mathrm{d}\hat\phi + 6\cos\tilde\theta\left(\mathrm{d}\tilde\phi + \frac{J}{\tilde\rho^2}\mathrm{d}t\right)\right)^2 \end{aligned} \tag{5.9}$$

with

$$Q = \frac{3}{2}C_2 \tag{5.10}$$

$$J = -C_1 \tag{5.11}$$

R-charge and Reeb vector. In order to correctly identify the R charge we proceed as in [27, 16, 28]. We define the new coordinates

$$\hat{\psi}' = \hat{\psi} \tag{5.12}$$

$$\beta = -6\alpha + c\hat{\psi} \tag{5.13}$$

In this coordinate system we can write the metric as a local $\mathrm{U}(1)$ fiber over a Kaehler-Einstein manifold and $\hat{\psi}'$ is a coordinate on the local $\mathrm{U}(1)$ fiber. From (2.17) of [16], we have

$$\mathrm{d}\Omega^2_{Y^{p,q}} = (e^{\hat{\theta}})^2 + (e^{\hat{\phi}})^2 + (e^{\hat{y}})^2 + (e^{\beta})^2 + (e^{\hat{\psi}})^2 \tag{5.14}$$

where the one forms on $Y^{p,q}$ are,

$$e^{\hat{\theta}} = \sqrt{\frac{1-c\hat{y}}{6}}d\hat{\theta}\,, \quad e^{\hat{\phi}} = \sqrt{\frac{1-c\hat{y}}{6}}\sin\hat{\theta}\mathrm{d}\hat{\phi}\,, \tag{5.15}$$

$$e^{\hat{y}} = \frac{1}{\sqrt{wq}}\mathrm{d}\hat{y}\,, \quad e^{\beta} = \frac{\sqrt{wq}}{6}(\mathrm{d}\beta + c\cos\hat{\theta}\mathrm{d}\hat{\phi})\,, \tag{5.16}$$

$$e^{\hat{\psi}'} = \frac{1}{3}(-\mathrm{d}\hat{\psi}' - \cos\hat{\theta}\mathrm{d}\hat{\phi} + \hat{y}(\mathrm{d}\beta + c\cos\hat{\theta}\mathrm{d}\hat{\phi}))\,. \tag{5.17}$$

As noted in [27], the R-symmetry is identified with a shift in the angular variable

$$\psi_R = -\frac{1}{2}\hat{\psi}' \tag{5.18}$$

at constant β. As a consequence, the $\mathrm{U}(1)$ R-symmetry gauge field is given by

$$A_R = -\frac{1}{2}A_Q \tag{5.19}$$

and

$$Q_R = -\frac{1}{2}Q\,. \tag{5.20}$$

The associated Killing vector is given by

$$K_R = -2\partial_{\hat{\psi}} - \frac{c}{3}\partial_\alpha \tag{5.21}$$

which coincides with the Reeb vector of the Sasaki-Einstein manifold. Notice that our Reeb vector differs by a factor of 2/3 from the one in [15, 16].

6. The 5-form and baryon charge

The self-dual Ramond-Ramond field strength $F_{(5)}$ can be written as

$$F_{(5)} = \mathcal{F}_5 + \star_{10}\mathcal{F}_5\,. \tag{6.1}$$

With our conventions and normalisations, the leading order for $\mathcal{F}_5$ is given by

$$\mathcal{F}^0_5 = L^4 Vol(Y^{p,q}) \tag{6.2}$$

where $Vol(Y^{p,q})$ is the volume form of the unit radius $Y^{p,q}$. The background metric is perturbed by the KK gauge fields as described in the previous section: the field strength is also perturbed in order to satisfy the equations of motion. The corrections are known to be of the form [27–29]

$$\mathcal{F}_5^1 = L^4 \mathrm{d}\left(A_Q \wedge \omega_Q + A_J \wedge \omega_J + A_B \wedge \omega_B\right) . \tag{6.3}$$

The $Y^{p,q}$ three forms ω_I are defined by

$$\mathrm{d}\omega_I + \iota_{K_I} Vol(Y^{p,q}) = 0 \tag{6.4}$$

where K_I, $I = J, Q$ is the Killing vector of $Y^{p,q}$ associated with the A_I gauge field. The 3-forms $\omega_{J,Q}$ are clearly defined up to the addition of a closed form. The 3-form ω_B is the generator of the one dimensional cohomology of the Sasaki-Einstein manifold and A_B is the gauge field dual to the baryon current of the CFT. The arbitrary shift by a closed form of the $\omega_{J,K}$ corresponds to the possibility of shifting the mesonic symmetries of the theory by an arbitrary baryonic one.

In the case of generic $Y^{p,q}$ for $c \neq 0$ we obtain the following form for the subleading corrections to $F_{(5)}$,

$$\begin{aligned}
\mathcal{F}_5^1 = &-\frac{2}{\tilde\rho^3}\mathrm{d}\tilde\rho \wedge \mathrm{d}t \wedge \Bigg\{ \frac{k}{4} \Bigg\langle \left(\sigma^{\hat 3} - 3\hat y(1-\hat y)\mathrm{d}\alpha\right)\sigma^{\hat 1}\wedge\sigma^{\hat 2} - \frac{3}{2(1-\hat y)^2}\sigma^{\hat 3}\wedge \mathrm{d}\alpha \wedge \mathrm{d}\hat y \Bigg] + \\
&+\frac{Q}{9}\left[\left(\frac{a-1}{3}\sigma^{\hat 3} - \frac{a-2\hat y(a-1)-3\hat y^2+2\hat y^3}{2(1-\hat y)}\mathrm{d}\alpha\right)\sigma^{\hat 1}\wedge\sigma^{\hat 2} + \frac{2+a-6\hat y+3\hat y^2}{4(1-\hat y)^2}\sigma^{\hat 3}\wedge\mathrm{d}\alpha\wedge\mathrm{d}\hat y\right] \\
&+\frac{J}{3}\left[\left(\frac{a-2\hat y+\hat y^2}{3}\sigma^{\hat 3} - \frac{a-2a\hat y+\hat y^2}{2(1-\hat y)}\mathrm{d}\alpha\right)\sigma^{\hat 1}\wedge\sigma^{\hat 2} - \frac{a-2\hat y+\hat y^2}{2(1-\hat y)^2}\sigma^{\hat 3}\wedge\mathrm{d}\alpha\wedge\mathrm{d}\hat y\right]\Bigg\} + \\
&+\frac{1}{\tilde\rho^2}\mathrm{d}t \wedge \left(-Q\frac{2(1-\hat y)}{9}\mathrm{d}\alpha - J\frac{4(1-\hat y)}{9}\sigma^{\hat 3}\right)\wedge \mathrm{d}\hat y \wedge \sigma^{\hat 1}\wedge\sigma^{\hat 2}
\end{aligned} \tag{6.5}$$

while for the case $c = 0$ and going to the natural coordinate $(\tilde\theta, \tilde\phi)$ defined by $(\hat y, \alpha) = (-\cos\tilde\theta, -\tilde\phi)$ we get

$$\begin{aligned}
\mathcal{F}_5^1 = -\frac{2}{\tilde\rho^3}\mathrm{d}\tilde\rho \wedge \mathrm{d}t \Bigg\{ &-\frac{k}{6}\left[\left(2\sigma^{\hat 3} - 6\cos\tilde\theta\mathrm{d}\tilde\theta\wedge\mathrm{d}\tilde\phi\right)\sigma^{\hat 1}\wedge\sigma^{\hat 2} - 3\sin\tilde\theta\sigma^{\hat 3}\wedge\mathrm{d}\tilde\theta\wedge\mathrm{d}\tilde\phi\right] \\
&+\frac{Q}{9}\left[\left(\frac{1}{3}\sigma^{\hat 3} - \cos\tilde\theta\mathrm{d}\tilde\phi\right)\sigma^{\hat 1}\wedge\sigma^{\hat 2} + \frac{1}{2}\sin\tilde\theta\sigma^{\hat 3}\wedge\mathrm{d}\tilde\theta\wedge\mathrm{d}\tilde\phi\right] \\
&+\frac{J}{3}\left[\left(-\frac{4}{3}\cos\tilde\theta\sigma^{\hat 3} + \frac{1}{2}(1+7\cos 2\tilde\theta)\mathrm{d}\tilde\phi\right)\sigma^{\hat 1}\wedge\sigma^{\hat 2} + \frac{1}{2}\sin 2\tilde\theta\sigma^{\hat 3}\wedge\mathrm{d}\tilde\theta\wedge\mathrm{d}\tilde\phi\right]\Bigg\} \\
+\frac{1}{\tilde\rho^2}\mathrm{d}t\wedge&\left(Q\frac{2}{9}\mathrm{d}\tilde\phi + J\frac{4}{9}\sigma^{\hat 3}\right)\wedge\sin\tilde\theta\mathrm{d}\tilde\theta\wedge\sigma^{\hat 1}\wedge\sigma^{\hat 2}
\end{aligned} \tag{6.6}$$

The volume form on $Y^{p,q}$ is given by[4]

$$\mathrm{Vol}(Y^{p,q}) = -e^{\hat y}\wedge e^{\beta}\wedge e^{\hat\theta}\wedge e^{\hat\phi}\wedge e^{\hat\psi'} = \frac{4(1-c\hat y)}{9}\mathrm{d}\hat y\wedge\mathrm{d}\alpha\wedge\sigma^{\hat 1}\wedge\sigma^{\hat 2}\wedge\sigma^{\hat 3}\,, \tag{6.7}$$

[4]The orientation is chosen to satisfy (6.2)

and we define the three forms

$$\omega_\pm \equiv e^{\hat\psi'} \wedge (e^{\hat\theta} \wedge e^{\hat\phi} \pm e^{\hat y} \wedge e^{\beta}) =$$
$$= \frac{1}{3}\left(2\sigma^{\hat 3}(1-c\hat y) - 6\hat y \mathrm{d}\alpha\right) \wedge \left(\frac{2(1-c\hat y)}{3}\sigma^{\hat 1}\wedge\sigma^{\hat 2} \mp \frac{1}{3}\mathrm{d}\hat y \wedge (c\sigma^{\hat 3} + 3\mathrm{d}\alpha)\right) \quad (6.8)$$

The local Kähler form J_2 is given by

$$J_2 = e^{\hat\theta}\wedge e^{\hat\phi} - e^{\hat y}\wedge e^{\beta} = \frac{1}{2}\mathrm{d}e^{\hat\psi'} \quad (6.9)$$

The closed form ω_B is given as in [28] by

$$\omega_B = \frac{9}{8\pi^2(1-c\hat y)^2}(p^2-q^2)\omega_- \quad (6.10)$$

With this normalisation and assuming that $A_B = \frac{Q_B}{\tilde\rho^2}\mathrm{d}t$, the baryon charge Q_B is given by

$$Q_B = \frac{\pi^2}{2(p^2-q^2)}k\,. \quad (6.11)$$

In the case of $T^{1,1}$ and recalling the change of coordinates $(\hat y, \alpha) = (-\cos\tilde\theta, -\tilde\phi)$ we get

$$\omega_\pm = \left(\frac{2}{3}\sigma^{\hat 3} - 2\cos\tilde\theta\mathrm{d}\tilde\phi\right)\wedge\left(\frac{2}{3}\sigma^{\hat 1}\wedge\sigma^{\hat 2} \pm \sin\tilde\theta \mathrm{d}\tilde\theta\wedge\mathrm{d}\tilde\phi\right) \quad (6.12)$$

with

$$\omega_B \equiv \frac{9}{8\pi^2}\omega_- \quad (6.13)$$

and

$$Q_B = \frac{2\pi^2}{3}k\,. \quad (6.14)$$

We now rewrite the expansion of $\mathcal{F}_5^1$ as

$$\mathcal{F}_5^1 = L^4\mathrm{d}\left(A_R\wedge\omega_R + A_\beta\wedge\omega_\beta + A_B\wedge\omega_B\right)\,. \quad (6.15)$$

where

$$A_R = -\frac{1}{2}A_Q, \qquad A_\beta = -6A_J - A_Q \quad (6.16)$$

are the gauge fields associated to the Killing vectors

$$K_R = -2\partial_{\hat\psi} - \frac{1}{3}\partial_\alpha, \qquad \partial_\beta = -\frac{1}{6}\partial_\alpha\,. \quad (6.17)$$

The remaining 3-forms are given by

$$\omega_R = -\frac{1}{6}\omega_+ \quad (6.18)$$
$$\omega_\beta = -\frac{(a-2\hat y+\hat y^2)}{18}\sigma^{\hat 3}\wedge\left(\frac{2}{3}\sigma^{\hat 1}\wedge\sigma^{\hat 2} - \frac{1}{2(1-\hat y)^2}\mathrm{d}\alpha\wedge\mathrm{d}\hat y\right) + \quad (6.19)$$
$$-\frac{a-2a\hat y+\hat y^2}{18(1-\hat y)}\mathrm{d}\alpha\wedge\sigma^{\hat 1}\wedge\sigma^{\hat 2}\,. \quad (6.20)$$

It can be shown without difficulty that they satisfy the expected relations

$$\mathrm{d}\omega_R + \iota_{(-2\partial_{\hat\psi}-\frac{1}{3}\partial_\alpha)}Vol(Y^{p,q}) = 0\,, \quad (6.21)$$
$$\mathrm{d}\omega_J + \iota_{\partial_\beta}Vol(Y^{p,q}) = 0\,. \quad (6.22)$$

7. Mass in asymptotically $AdS_5 \times X^5$

In this section we derive the expression for the mass in the asymptotically $AdS_5 \times Y^{p,q}$ spacetimes under examination. There has been a considerable amount of work over the years on the definition of mass and other conserved charges in general relativity. The issue becomes even subtler in the case of the definition of the mass in asymptotically AdS spaces. For example, the standard expression given in terms of a Komar integral gives a divergent result in this case, and the procedure of renormalisation is ambiguous. We will follow the definition of conserved charges given by Wald and collaborators [26, 25] which provides a possible general framework for addressing this issue, and apply it to our case for the computation of the mass. Since our solutions mix beyond the leading order AdS and $Y^{p,q}$ coordinates, it is natural to take a ten dimensional approach for the definition of mass, which has the advantage of being relatively simple both from the conceptual and from the technical point of view.

Another derivation of conserved charges applicable to spacetime with AdS asymptotics (more precisely asymptotically locally AdS spacetimes) was presented in [30]. Using non linear KK mapping one can also uplift this derivation to ten dimensional asymptotically $AdS_5 \times X^5$ backgrounds [31, 32].

The main result of this section is to prove that, with the adopted definition of mass, the expected BPS relation:

$$ML = \frac{3}{2}R \tag{7.1}$$

which is a consequence of the $\mathcal{N} = 1$ superconformal algebra on the field theory side, is satisfied.

7.1 Definition of charges in asymptotically $AdS_5 \times X^5$

We are dealing with an asymptotically $AdS_5 \times X^5$ spacetime, where X^5 is a compact manifold.

It is convenient to choose coordinates such that, defining a radial AdS coordinate Ω, $g_{\Omega\Omega} = L^2/\Omega^2$ and $g_{\Omega M} = 0$ for $M \neq \Omega$, M denoting a ten dimensional coordinate. We will also denote the AdS coordinates with $\mu, \nu, \ldots$ and the internal coordinates with $a, b, \ldots$. At leading order for large Ω we have

$$\mathrm{d}s^2 = \frac{L^2}{\Omega^2}\left[\mathrm{d}\Omega^2 - \mathrm{d}t^2 + \mathrm{d}\Omega_3^2\right] + L^2\mathrm{d}s^2(Y^{p,q}) \,. \tag{7.2}$$

We will keep corrections to orders Ω^{2k}, with $k = 0, 1, 2$ for the AdS part, $g_{\mu\nu}$, $k = 1, 2$ for the internal, g_{ab}, and mixed parts, $g_{\mu a}$ respectively. There are of course corrections of higher order in Ω to the background 5-form given by the volume forms on AdS_5 and $Y^{p,q}$ which we will discuss later.

In general the construction of conserved charges proceeds as follows: let us denote for the moment as φ the generic field appearing in a Lagrangian $\mathcal{L}$. The variation of the Lagrangian with respect to φ is given by

$$\delta\mathcal{L} = E(\varphi)\delta\varphi + \mathrm{d}\theta(\varphi, \delta\varphi) \,. \tag{7.3}$$

where $E(\varphi)$ denotes the equations of motion. This defines the symplectic potential θ, corresponding to the boundary term that arises from integrating by parts in order to remove derivative of $\delta\varphi$. It is a 9-form in spacetime.

We will be interested in the following asymptotic symmetry generator

$$\xi = \frac{\partial}{\partial t} \tag{7.4}$$

We want to identify the Hamiltonian generator $\mathcal{H}_\xi$ of such symmetry. Its value on the desired solution will be our definition of the mass of the metric [5] $\mathcal{H}_\xi$ is defined via its variation with respect to a generic fluctuation $\delta\phi$, obeying the linearised equations of motion in a given background obeying the full equations of motion [26]:

$$\delta\mathcal{H}_\xi = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) \tag{7.5}$$

where Σ is a 9 dimensional submanifold of the spacetime without boundary, a "slice" corresponding to the vector field ξ. By the integral over $\partial\Sigma$ we mean a limiting process in which the integral is first taken over the boundary ∂K of a compact region inside Σ and then K approaches Σ in a suitable manner. The 8-form Q_ξ is the Noether charge of the asymptotic symmetry ξ. It has a contribution coming from the gravitational lagrangian:

$$Q^{\rm grav}_{\alpha_1\cdots\alpha_8} = -\frac{1}{16\pi G_{10}}\nabla^b\xi^c\epsilon_{bca_1\cdots a_8}\,. \tag{7.6}$$

where $\epsilon = \sqrt{-\det g}\;\mathrm{d}^{10}x$ is the volume form. Also, the gravitational contribution to θ is:

$$\theta^{\rm grav}_{a_1\cdots a_9} = \frac{1}{16\pi G_{10}} v^a\epsilon_{aa_1\cdots a_9} \tag{7.7}$$

with

$$v^a = \nabla^b\delta g_b{}^a - \nabla^a\delta g_b{}^b \tag{7.8}$$

Finally, the RR 5-form contributes both to Q_ξ and θ, giving rise to a single term in the combination $\delta Q_x i - \xi\cdot\theta$. With our normalisation for the 5-form F_5, the final result for $\delta\mathcal{H}_x i$ is

$$\delta\mathcal{H}_\xi = \int_{\partial\Sigma}\frac{1}{16\pi G_{10}}\left(-\delta Q^{\rm grav}_\xi - \xi^{a1}\left(v^a\epsilon_{aa_1\cdots a_9} - 128F_{a_1\cdots a_5}\delta A_{a_6\cdots a_9}\right)\right) \tag{7.9}$$

where $F^{(5)} = \mathrm{d}A^{(4)}$.

Under mild assumptions [26], a necessary and sufficient condition for the existence of $\mathcal{H}_\xi$ is the integrability of the equation for $\mathcal{H}_\xi$:

$$(\delta_1\delta_2 - \delta_2\delta_1)\mathcal{H}_\xi = 0 \tag{7.10}$$

i.e.

$$0 = \xi\cdot[\delta_2\theta(\varphi,\delta_1\varphi) - \delta_1\theta(\varphi,\delta_2\varphi)] \tag{7.11}$$

[5] We are specifying here to a particular symmetry generator since we are interested in the mass, but the same procedure con be applied to the most general asymptotic symmetry generator [26].

When this condition is satisfied we are guaranteed that an 8-form I_ξ exists whose variation is

$$\delta I_\xi = \delta Q_\xi - \xi \cdot \theta \tag{7.12}$$

The value of the global charge associated with the asymptotic isometry generated by ξ is given by a simple "surface" integral, up to an arbitrary constant which can be determined by fixing the value of the charge for a "reference solution".

$$\mathcal{H}_\xi = \int_{\partial\Sigma} I_\xi + \mathcal{H}^0_\xi. \tag{7.13}$$

Notice that the 8-manifold $\partial\Sigma$ in the present case reduces asymptotically to $S^3 \times Y_{pq}$, where S^3 is a 3-sphere of radius L/Ω inside AdS_5. The existence of $\mathcal{H}$ can be explicitly checked for a background with the asymptotic behaviour we have discussed above for the metric. The expression for $\theta^{\rm grav}$ in our gauge is proportional to

$$\xi \cdot \theta^{\rm grav}(\delta g) = \left(\Omega^2 \delta(g^{tM}\partial_\Omega g_{aM}\sqrt{g}) - g^{MN}\sqrt{g}(\partial_\Omega \delta g_{MN} - \Gamma^P_{\Omega M}\delta g_{PN})\right)\epsilon_{t\Omega M_1\ldots M_8} \tag{7.14}$$

One can verify, using the asymptotic expansion for the metric given before, that $\delta_{[1}\theta(\delta_{2]}g) = 0$. The crucial fact for this result to hold is that $g^{MN}\delta g_{MN} = \mathcal{O}(\Omega^4)$. This is satisfied by our BPS solutions, but can be proven to hold more generally, even for non necessarily BPS solutions of the equations of motion, given an appropriate asymptotic behaviour [33]. One can similarly verify that the contribution of the 5-form to θ is integrable.

Once we have verified the existence of $\mathcal{H}_\xi$, we can define the mass of a generic solution $\mathcal{M}$ to the equations of motion as the value of $\mathcal{H}_\xi$ on such a solutions

$$M_{\mathcal{M}} \equiv H_\xi|_{\mathcal{M}} \,. \tag{7.15}$$

7.2 Calculation of mass and R-charge

We will now proceed to the calculation of the mass and R-charge for the solutions we have described in the previous sections. We are interested in the dependence of the mass $M =$ on the integration constants, C_1, C_3 and k. Therefore we will compute, $\frac{\partial M}{\partial C_i}$ and $\frac{\partial M}{\partial k}$, by plugging in (7.9) the expressions for the background given by our solutions.

Using the expressions for the leading order, first and second subleading orders for the metric and the 5-form given in 5 and in the appendix one can calculate

$$\begin{aligned}
\frac{\partial M}{\partial k} &= \int_{S^3\times Y^{p,q}} \left(\frac{\partial}{\partial k} Q^{\rm grav}_\xi - \xi\cdot\theta^k\right) = 0 \\
\frac{\partial M}{\partial C_1} &= \int_{S^3\times Y^{p,q}} \left(\frac{\partial}{\partial C_1} Q^{\rm grav}_\xi - \xi\cdot\theta^1\right) = 2\frac{\pi L^2}{4G_5} \\
\frac{\partial M}{\partial C_3} &= \int_{S^3\times Y^{p,q}} \left(\frac{\partial}{\partial C_2} Q^{\rm grav}_\xi - \xi\cdot\theta^2\right) = -3\frac{\pi L^2}{4G_5}
\end{aligned} \tag{7.16}$$

where G_5 is the 5-dimensional Planck constant $G_5 = G_{10}/Vol(Y^{p,q})$ and

$$\theta^i_{a_1\cdots a_9} = \frac{1}{16\pi G_{10}}\left[\left(\nabla^b\partial_i g_b{}^a - \nabla^a\partial_i g_b{}^b\right)\epsilon_{aa_1\cdots a_9} - 128 F_{a_1\cdots a_5}\partial_i A_{a_6\cdots a_9}\right] \tag{7.17}$$

with

$$\partial_i = \frac{\partial}{\partial k}, \frac{\partial}{\partial C_i}. \tag{7.18}$$

Putting everything together we conclude that:

$$M = \frac{\pi L^2}{4G_5}(2C_1 - 3C_2) = -\frac{\pi L^2 Q}{4G_5}, \tag{7.19}$$

where we have set the integration constant to zero. Some comments are in order here. First note that the 8-form to be integrated involves directions orthogonal to t and Ω. The relevant contribution from the 5-form is of the type $F^{(5)}_{t\hat{y}\tilde{1}\tilde{2}\tilde{3}}\partial_i A^{(4)}_{\phi\hat{1}\hat{2}\hat{3}}$, which turns out to be of order Ω^0: $\partial_i A^{(4)}_{\phi\hat{1}\hat{2}\hat{3}}$ goes like Ω^2, and $F^{(5)}_{\Omega\phi\hat{1}\hat{2}\hat{3}} = \partial_\Omega A^{(4)}_{\phi\hat{1}\hat{2}\hat{3}} \sim \Omega$. On the other hand, $F^{(5)}_{t\hat{y}\tilde{1}\tilde{2}\tilde{3}}$, the dual of the latter, goes like Ω^{-2}. Therefore the 5-form term gives a finite contribution to $\partial_i M$. The gravitational contributions to $\partial_i M$ on the other hand contain terms of order $1/\Omega^2$, therefore potentially divergent. However the coefficients of these terms turn out to be total derivatives in the internal coordinates: more precisely, the coefficient is proportional to $\frac{\mathrm{d}}{\mathrm{d}\hat{y}}q(\hat{y})$, therefore it gives vanishing contribution after integrating over $\hat{y}$ between the two zeroes of $q(\hat{y})$, $\hat{y}_1$ and $\hat{y}_2$. Again this fact can be proven in more generality than just for our BPS solutions [33].

Let us now proceed to verify the BPS relation between the mass M and the R-charge R. With our normalisation of the Reeb vector, the BPS relation is given by

$$ML = \frac{3}{2}R \tag{7.20}$$

where R is the charge which sources the KK gauge field A_R. The five dimensional equation of motion for its field strength F^R are given by

$$\tau_{RR}\, \mathrm{d} \star_5 F^R = \star_5 J^R\,. \tag{7.21}$$

where J^R is the one-form charge current and τ_{RR} comes from the KK reduction. Taking the integral of the equation of motion, the total charge R can be read from the flux at infinity of the field strength

$$R = \lim_{\tilde{\rho}\to\infty} \tau_{RR} \int_{S^3(\tilde{\rho})} F^R \tag{7.22}$$

where $S^3(\tilde{\rho})$ is the three dimensional sphere in AdS_5 at constant $t, \tilde{\rho}$. In section 5 we derived

$$A^R \approx -\frac{Q}{2\tilde{\rho}^2}\mathrm{d}t \tag{7.23}$$

at leading order in large $\tilde{\rho}$. Following [34] we have

$$\tau_{RR} = \frac{3}{16\pi G_{10}} \int g_{\psi_R\psi_R} vol(Y^{p,q}) = \frac{1}{12\pi G_5} \tag{7.24}$$

where we have used $g_{\psi_R\psi_R} = \frac{4}{9}$ as can be seen from (5.17),(5.18). We can now explicitly write the value of the total R charge

$$R = -\frac{QL^3}{12\pi G_5} Vol(S^3) = \frac{2}{3}ML \tag{7.25}$$

which satisfies the expected relation.

Let us mention that we have computed M also using a 5-dimensional definition involving the intrinsic 5-dimensional Weyl tensor, due to Ashtekar and collaborators [24] and rederived in [25],

$$\mathcal{H}_\xi = M = -\frac{1}{8\pi G_5}\int_{S^3} \tilde{E}_{tt} vol(S^3) \tag{7.26}$$

where

$$\tilde{E}_{tt} = \frac{1}{2}\Omega^{-2}\tilde{C}_{\Omega tt\Omega}\,. \tag{7.27}$$

where $\tilde{C}_{abcd}$ is the Weyl tensor of the unphysical metric $\tilde{g} = \Omega^2 g$. Beyond leading order AdS_5 and $Y^{p,q}$ coordinates mix, so, in general the metric on the deformed AdS_5 depends on the choice of the 5-dimensional slice inside the 10-dimensional manifold. The calculation, done using our explicit form of the perturbed metric and allowing a slice dependence on the internal coordinates, actually reveals that the slice dependence drops out in the Weyl tensor and gives the correct result for the mass, as in the previous 10-dimensional computation. The degree of generality of this result is under investigation [33].

8. Conclusions and open problems

In this paper we have performed an asymptotic, large distance, analysis of 1/2 BPS states in IIB supergravity $AdS_5 \times Y^{p,q}$. The corresponding differential equations are the same as those found in [12], where 1/8 BPS states of IIB supergravity on $AdS_5 \times S^5$ were analysed. The difference resides in the boundary conditions, here we require solutions which are asymptotic to $AdS_5 \times Y^{p,q}$. They carry non trivial charges under the asymptotic isometries which are dual to the R-charge and one U(1) flavour charges of the quiver gauge theories. We have shown that the charges are consistent with the holographic principle which in this case relates $\mathcal{N} = 1$ quiver gauge theories to gravity on $AdS_5 \times Y^{p,q}$. Of course our analysis was only asymptotic: we did not address the issue of regularity of the solutions over the full configuration space. One can analyse the solutions in the opposite regime, near $y = 0$, but it is difficult to connect this region to the large y region. It would be very interesting, although probably quite hard, due to the complexity of the system of non-linear partial differential equations governing them, to prove the existence of non-singular solutions, which then would be the exact analog of those found in [1] for the maximally supersymmetric case.

In the course of the analysis we had to cope with the problem of defining the mass of the states in the asymptotically $AdS_5 \times Y^{p,q}$ spacetime. We adopted a ten dimensional approach, which uses the definition of charges given by Wald and collaborators. It gives a finite (and correct) result. A different "holographic" approach to this problem, which uses a detailed analysis of the KK reduction of the 10 dimensional theory to AdS_5 can be found in [30 − 32]. We had indications, however, that, at least for our backgrounds, an expression due to Ashtekar and Das [24, 25], which involves the intrinsic Weyl tensor in the deformed AdS_5, also gives the correct result. This brings about various questions. For example, about the finiteness of Wald et al. expression, one would like to establish it in general terms,

JHEP02(2008)050

without relying to a particular form of backgrounds. That is, one would like to prove in general, assuming just that the equations of motion hold with the asymptotic behaviour of the fields implied by AdS/CFT correspondence, that potentially divergent terms are total derivatives in the internal compact manifold. Similarly, it would be interesting to see under which circumstances the ten dimensional approach finally coincides with the 5-dimensional one of Ashtekar et al. We hope to come back to these issues in a future publication [33].

A. Second order solutions

We give here the complete expression for the second order solutions

$$
\begin{aligned}
\rho_1^{(2)}(\hat{y}) = &-(L^8(-1+c\hat{y})^2((-4+4ac^2+27k)^2(-80+496c\hat{y}-584\ c^2\hat{y}^2-2696c^3\hat{y}^3+11666c^4\hat{y}^4\\
&-19494c^5\hat{y}^5+16281c^6\hat{y}^6-6696c^7\hat{y}^7+1080c^8\hat{y}^8+a^3c^6(-65+72c\hat{y}+20c^2\hat{y}^2)+\\
&+a^2c^4(50+82c\hat{y}+159c^2\hat{y}^2-752c^3\hat{y}^3+380c^4\hat{y}^4)+ac^2(-40-56c\hat{y}+756c^2\hat{y}^2\\
&-3572c^3\hat{y}^3+6449c^4\hat{y}^4-4536c^5\hat{y}^5+1080c^6\hat{y}^6))+\\
&-8(-1+ac^2)(-4+4ac^2+27k)(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2(-20+c\hat{y}(4-84C_1)\\
&-264c^3\hat{y}^3(1+4C_1)+120c^4\hat{y}^4(1+4C_1)+c^2\hat{y}^2(133+552C_1)\\
&+ac^2(5-60C_1+20c^2\hat{y}^2(1+3C_1)+2c(\hat{y}+54\hat{y}C_1)))\\
&+32(-1+ac^2)^2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2(-10+c\hat{y}(2-84C_1)\\
&+10c^4\hat{y}^4(-1+10C_1+27C_1^2)+ac^2(-2c\hat{y}(17+41C_1)+10c^2(\hat{y}+3\hat{y}C_1)^2\\
&+5(3+8C_1-6C_2))-2c^3\hat{y}^3(1+143C_1+270C_1^2+30C_2)\\
&+c^2\hat{y}^2(29+252C_1+180C_1^2+90C_2))))/(4320(-1+ac^2)^2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^3)
\end{aligned}
$$

$$
\rho_3^{(2)}(\hat{y}) = \rho_1^{(2)}(\hat{y}) + k^{(2)}(\hat{y})
$$

$$
\begin{aligned}
k^{(2)}(\hat{y}) = &(L^6(L-cL\hat{y})^2(((1-c\hat{y})(16(-1+ac^2)^2(-1+c\hat{y})(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2\\
&(11+4ac^2-30c\hat{y}+15c^2\hat{y}^2)-4(-1+ac^2)(-4+4ac^2+27k)(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)\\
&(-44+284c\hat{y}-576c^2\hat{y}^2+426c^3\hat{y}^3-90c^4\hat{y}^4-9c^5\hat{y}^5+a^2c^4(-14+5c\hat{y})\\
&+2ac^2(-34+103c\hat{y}-72c^2\hat{y}^2+12c^3\hat{y}^3))+(-4+4ac^2+27k)^2(a^3c^6(11+c\hat{y})\\
&+6a^2c^4(-17+25c\hat{y}-21c^2\hat{y}^2+7c^3\hat{y}^3)+3ac^2(-36+220c\hat{y}-324c^2\hat{y}^2+200c^3\hat{y}^3\\
&-51c^4\hat{y}^4+3c^5\hat{y}^5)+2(-22+202c\hat{y}-666c^2\hat{y}^2+894c^3\hat{y}^3-531c^4\hat{y}^4\\
&+117c^5\hat{y}^5))))/((1-ac^2)(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2)\\
&-12c(4a^2c^3(-4-8C_1+7c(\hat{y}+2\hat{y}C_1)+6C_2)+\hat{y}(27k(2-8c\hat{y}+3c^3\hat{y}^3)(1+2C_1)\\
&-4c\hat{y}(-3+2c\hat{y})(-4-8C_1+7c(\hat{y}+2\hat{y}C_1)+6C_2))+ac(4c(-7+27k)\hat{y}(1+2C_1)\\
&+56c^4\hat{y}^4(1+2C_1)-4c^3\hat{y}^3(29+58C_1-12C_2)+8(2+4C_1-3C_2)\\
&-3c^2\hat{y}^2(-16+9k-32C_1+18kC_1+24C_2))))))/(648(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2)
\end{aligned}
$$

$$
\begin{aligned}
t^{(2)}(\hat{y}) = {} & (L^8(-1+c\hat{y})^2(2187k^2(-1+c\hat{y})^2(-2+14c\hat{y}-9c^2\hat{y}^2+ac^2(-7+4c\hat{y})) \\
& -216(-1+ac^2)k(-1+c\hat{y})(2+2a^2c^4+27c^2\hat{y}^2-45c^3\hat{y}^3 \\
& +18c^4\hat{y}^4+ac^2(-13+9c\hat{y}))+16(-1+ac^2)^2(-9(-1+c\hat{y})^2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2) \\
& +8(-1+c\hat{y})(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2C_1+1/1-ac^2(3(-(-1+ac^2)^3C_2 \\
& -3(-1+ac^2)^2(-1+c\hat{y})^2C_2+27(-1+c\hat{y})^6C[3] \\
& -(-1+ac^2)(-1+c\hat{y})^3(-4+4ac^2+(9-9c\hat{y})C_2))))))/(216(1-ac^2) \\
& (2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^3)
\end{aligned}
$$

References

[1] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and* 1/2 *BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174].

[2] N.V. Suryanarayana, *Half-BPS giants, free fermions and microstates of superstars*, *JHEP* **01** (2006) 082 [hep-th/0411145].

[3] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075].

[4] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015].

[5] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016].

[6] J.T. Liu, D. Vaman and W.Y. Wen, *Bubbling* 1/4 *BPS solutions in type IIB and supergravity reductions on* $S^n \times S^n$, *Nucl. Phys.* **B 739** (2006) 285 [hep-th/0412043].

[7] Z.W. Chong, H. Lü and C.N. Pope, *BPS geometries and AdS bubbles*, *Phys. Lett.* **B 614** (2005) 96 [hep-th/0412221].

[8] A. Donos, *A description of* 1/4 *BPS configurations in minimal type IIB SUGRA*, *Phys. Rev.* **D 75** (2007) 025010 [hep-th/0606199].

[9] A. Donos, *BPS states in type IIB SUGRA with* $\mathrm{SO}(4)\times\mathrm{SO}(2)_{\mathrm{gauged}}$ *symmetry*, *JHEP* **05** (2007) 072 [hep-th/0610259].

[10] N. Kim, *AdS_3 solutions of IIB supergravity from D3-branes*, *JHEP* **01** (2006) 094 [hep-th/0511029].

[11] B. Chen et al., *Bubbling AdS and droplet descriptions of BPS geometries in IIB supergravity*, *JHEP* **10** (2007) 003 [arXiv:0704.2233].

[12] E. Gava, G. Milanesi, K.S. Narain and M. O'Loughlin, 1/8 *BPS states in AdS/CFT*, *JHEP* **05** (2007) 030 [hep-th/0611065].

[13] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, *Int. J. Mod. Phys.* **A 22** (2007) 1353 [hep-th/0601089].

[14] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026 [hep-th/0604133].

[15] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on* $S^2\times S^3$, *Adv. Theor. Math. Phys.* **8** (2004) 711 [hep-th/0403002].

JHEP02(2008)050

[16] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, *Commun. Math. Phys.* **262** (2006) 51 [hep-th/0411238].

[17] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064 [hep-th/0411264].

[18] D. Arean, D.E. Crooks and A.V. Ramallo, *Supersymmetric probes on the conifold*, *JHEP* **11** (2004) 035 [hep-th/0408210].

[19] F. Canoura, J.D. Edelstein, L.A.P. Zayas, A.V. Ramallo and D. Vaman, *Supersymmetric branes on $AdS_5 \times Y^{(p,q)}$ and their field theory duals*, *JHEP* **03** (2006) 101 [hep-th/0512087].

[20] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an $N = 1$ superconformal gauge theory*, *Phys. Rev.* **D 58** (1998) 125025 [hep-th/9808075].

[21] D. Martelli and J. Sparks, *Dual giant gravitons in Sasaki-Einstein backgrounds*, *Nucl. Phys.* **B 759** (2006) 292 [hep-th/0608060].

[22] A. Basu and G. Mandal, *Dual giant gravitons in $AdS_m \times Y^n$ (Sasaki-Einstein)*, *JHEP* **07** (2007) 014 [hep-th/0608093].

[23] A. Buchel and J.T. Liu, *Gauged supergravity from type IIB string theory on $Y^{(p,q)}$ manifolds*, *Nucl. Phys.* **B 771** (2007) 93 [hep-th/0608002].

[24] A. Ashtekar and S. Das, *Asymptotically anti-de Sitter space-times: conserved quantities*, *Class. and Quant. Grav.* **17** (2000) L17 [hep-th/9911230].

[25] S. Hollands, A. Ishibashi and D. Marolf, *Comparison between various notions of conserved charges in asymptotically AdS-spacetimes*, *Class. and Quant. Grav.* **22** (2005) 2881 [hep-th/0503045].

[26] R.M. Wald and A. Zoupas, *A general definition of "Conserved quantities" in general relativity and other theories of gravity*, *Phys. Rev.* **D 61** (2000) 084027 [gr-qc/9911095].

[27] D. Berenstein, C.P. Herzog and I.R. Klebanov, *Baryon spectra and AdS/CFT correspondence*, *JHEP* **06** (2002) 047 [hep-th/0202150].

[28] C.P. Herzog, Q.J. Ejaz and I.R. Klebanov, *Cascading RG flows from new Sasaki-Einstein manifolds*, *JHEP* **02** (2005) 009 [hep-th/0412193].

[29] S. Benvenuti, L.A. Pando Zayas and Y. Tachikawa, *Triangle anomalies from Einstein manifolds*, *Adv. Theor. Math. Phys.* **10** (2006) 395 [hep-th/0601054].

[30] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [hep-th/0505190].

[31] K. Skenderis and M. Taylor, *Kaluza-Klein holography*, *JHEP* **05** (2006) 057 [hep-th/0603016].

[32] K. Skenderis and M. Taylor, *Anatomy of bubbling solutions*, *JHEP* **09** (2007) 019 [arXiv:0706.0216].

[33] E. Gava, G. Milanesi, K. Narain and M. O'Loughlin, in preparation.

[34] E. Barnes, E. Gorbatov, K. Intriligator and J. Wright, *Current correlators and AdS/CFT geometry*, *Nucl. Phys.* **B 732** (2006) 89 [hep-th/0507146].

JHEP02(2008)050